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Luminosity Formulas for Asymmetric Colliders with Beam Symmetries

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Summary

We summarize the (mostly known) formulas for the luminosity of an asymmetric collider when the beams obey the "transparency symmetry," the "horizontal–vertical" symmetry, or both symmetries combined.

1. Nominal and dynamical beam quantities

In the absence of the beam-beam interaction, the beam parameters are determined by the lattice, the energy and rf parameters of each of the collider's rings. In particular, this is true of the emittances and therefore of the beam sizes at the interaction point. From these one can compute the beam-beam parameters and the luminosity in the weak-beam limit; the quantities calculated in this limit are called "nominal," and are identified by a subscript 0. For example, the nominal vertical beam size at the interaction point $\sigma^*_{0y,+}$ and beam-beam parameter $\xi_{0y,+}$ of the e^+ beam, and the nominal luminosity $\mathscr{L}_0$ are given by

$$
\begin{aligned}
\sigma^*_{0y,+} &= \sqrt{\varepsilon_{0y,+}\beta^*_{y+}} \\
\xi_{0y,+} &= \frac{r_0 N_- \beta^*_{y+}}{2\pi\gamma_+\sigma^*_{0y,-}(\sigma^*_{0x,-}+\sigma^*_{0y,-})} \\
\mathscr{L}_0 &= \frac{N_+ N_- f_c}{2\pi\sqrt{\left(\sigma^{*2}_{0x,+}+\sigma^{*2}_{0x,-}\right)\left(\sigma^{*2}_{0y,+}+\sigma^{*2}_{0y,-}\right)}}
\end{aligned}
\tag{1.1}
$$

where $\beta^*_{y,+}$ and $\varepsilon_{0y,+}$ are the vertical beta-function at the interaction point and nominal emittance of the e^+ beam, the $N_\pm$are the number of particles per bunch, r_0 is the classical electron radius, γ_+ is the usual relativistic factor and f_c is the bunch collision frequency. If the bunches are evenly spaced by a distance s_B, the bunch collision frequency is, in the relativistic limit, $f_c=c/s_B$, c being the speed of light. There are three more beam-beam parameters, whose expressions can be obtained from the above by the replacements $x \leftrightarrow y$ and/or $+ \leftrightarrow -$.

The beam-beam interaction induces tune shifts in both beams. In the limit when the ξ_0 parameters are small and the base tunes are not too close to the integer, which we assume to be the case, the tune shift equals ξ_0. There are four (in general, different) tune shifts; the labeling we have chosen for the beam-beam parameters shown above is such that the subscripts correspond to these tune shifts. For example, $\xi_{0y,+}$ equals the vertical tune shift of the e^+ beam.

The above formulas are valid under the assumptions that: (a) the bunches collide head-on; (b) the bunches have elliptical gaussian transverse profile with common axes; (c) the bunches have lengths small compared to their transverse size; (d) the beam-beam interaction does not induce coherent oscillations or a relative displacement of the closed orbits at the interaction point. Under these assumptions the above formula for the luminosity is valid for arbitrary transverse beam sizes and beta-functions.

Once the beams are brought into collision the emittances inevitably deviate from their nominal values and, as a result, so do all quantities involving the beam sizes, such as the luminosity. These are the "dynamical" quantities, denoted without the subscript 0; the dynamical quantities corresponding to the above are given by

$$
\begin{aligned}
\sigma^*_{y,+} &= \sqrt{\varepsilon_{y,+}\beta^*_{y+}} \\
\xi_{y,+} &= \frac{r_0\, N_-\, \beta^*_{y+}}{2\pi\gamma_+\sigma^*_{y,-}(\sigma^*_{x,-}+\sigma^*_{y,-})} \qquad (1.2) \\
\mathcal{L} &= \frac{N_+\, N_-\, f_c}{2\pi\sqrt{\left(\sigma^{*2}_{x,+}+\sigma^{*2}_{x,-}\right)\left(\sigma^{*2}_{y,+}+\sigma^{*2}_{y,-}\right)}}
\end{aligned}
$$

The beam symmetries described below are effected by imposing constraints on the nominal quantities. If the symmetry is perfect, it will survive when the beams are brought into collision, except for the possibility of spontaneous symmetry breaking. If this breaking does not occur, the dynamical quantities will, of course, also obey the symmetry.

2. Beam symmetries

The fact that an asymmetric collider necessarily consists of two rings enlarges the beam dynamics parameter space considerably relative to a single-ring, symmetric, collider. The bunches in the two rings see different rf systems, different lattice functions and different magnet errors. Even the simplest beam-beam dynamics study requires, at a minimum, the specification of two values for the number of particles per bunch $N_\pm$, six beam sizes (two transverse and one longitudinal for each beam), four beta-functions at the IP (one vertical and one horizontal for each beam), six tunes and two damping decrements, not to mention the two energies.

Because no asymmetric colliders are in existence, and because the consequences of the beam-beam interaction are not completely understood for intense beams, it has been argued [1] that a cautious approach to an asymmetric collider design is to force the beam dynamics of an asymmetric collider to resemble as closely as possible that of a symmetric one. In this way one might be guided by the experience available from single-ring colliders. This is the so-called "transparency symmetry;" it is achieved by imposing the following constraints on the parameters of the two rings:

(1) pairwise equality of the nominal beam-beam parameters, $\xi_{0x,+}=\xi_{0x,-}$, $\xi_{0y,+}=\xi_{0y,-}$;

(2) pairwise equality of nominal beam sizes, $\sigma^*_{0x,+}=\sigma^*_{0x,-}$, $\sigma^*_{0y,+}=\sigma^*_{0y,-}$;

(3) equality of damping decrements of the two rings; and

(4) equality of the tune modulation amplitudes due to synchrotron oscillations, $(\sigma_s \nu_s/\beta^*_x)_+=(\sigma_s \nu_s/\beta^*_x)_-$, $(\sigma_s \nu_s/\beta^*_y)_+=(\sigma_s \nu_s/\beta^*_y)_-$ where σ_s is the bunch length and ν_s the synchrotron tune.

An immediate consequence of the transparency symmetry is a significant reduction in the number of free parameters, which is certainly of practical advantage for beam-beam studies. However, it has been argued that [2], given an asymmetric machine design, the beam-beam limit (maximum luminosity with acceptable beam lifetime) subject to certain constraints, such as cost, can, in general, only be achieved with asymmetric beam dynamics parameters. It is possible, however, that achieving this beam-beam limit entails a large sensitivity to beam or lattice parameters and therefore undesirably tight tolerances in the operation of the machine. These are, of course, matters that deserve careful investigation.

Another kind of symmetry that is possible for an asymmetric collider is the "horizontal-vertical" symmetry, defined by replacing condition (1) above by the following

(1´) $\xi_{0x,+}=\xi_{0y,+}$, $\xi_{0x,-}=\xi_{0y,-}$

while maintaining condition (2) (conditions (3) and (4) may be different from the transparency case, but this is not important for the purposes of this note). This symmetry also has the practical consequence of reducing the parameter space of the collider.

Both symmetries imply useful formulas connecting the luminosity to the beam-beam parameters, which we will derive in turn below. In either case, however, condition (2) implies that there is a single nominal aspect ratio r,

$$\left(\frac{\sigma^*_{0y}}{\sigma^*_{0x}}\right)_+ = \left(\frac{\sigma^*_{0y}}{\sigma^*_{0x}}\right)_- \equiv r \tag{2.1}$$

and that the expression for the nominal luminosity simplifies to

$$\mathcal{L}_0 = f_c \frac{N_+ N_-}{4\pi\sigma^*_{0x}\sigma^*_{0y}} \tag{2.2}$$

We now derive the luminosity formulas for the two cases separately.

<u>(A) Transparency Symmetry</u>

By imposing condition (1), $\xi_{0x,+}=\xi_{0x,-}$, $\xi_{0y,+}=\xi_{0y,-}$, one immediately finds

$$\frac{N_-\beta^*_{x,+}}{\gamma_+} = \frac{N_+\beta^*_{x,-}}{\gamma_-} \quad \text{and} \quad \frac{N_-\beta^*_{y,+}}{\gamma_+} = \frac{N_+\beta^*_{y,-}}{\gamma_-} \tag{2.3}$$

or, equivalently,

$$\left(\frac{EI}{\beta^*_x}\right)_+ = \left(\frac{EI}{\beta^*_x}\right)_- \quad \text{and} \quad \left(\frac{EI}{\beta^*_y}\right)_+ = \left(\frac{EI}{\beta^*_y}\right)_- \tag{2.4}$$

(we assume that there are no gaps in the beam, so that the total current is $I_\pm = e\, f_c\, N_\pm$). By dividing these equations in pairs one finds that there is a single (rather than two) ratio of beta-functions

$$\left(\frac{\beta^*_y}{\beta^*_x}\right)_+ = \left(\frac{\beta^*_y}{\beta^*_x}\right)_- \equiv r_\beta \tag{2.5}$$

and also

$$\frac{\beta^*_{x,-}}{\beta^*_{x,+}} = \frac{\beta^*_{y,-}}{\beta^*_{y,+}} = \frac{(EI)_-}{(EI)_+} \tag{2.6}$$

Now, by using (2.5), the definition of σ^*_0 and (2.1) one finds that there is also a single emittance ratio,

$$\left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_+ = \left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_- \equiv r_\varepsilon \tag{2.7}$$

so that

$$r = \sqrt{r_\varepsilon r_\beta} \tag{2.8}$$

From the definition of ξ_0, one finds the ratio

$$\frac{\xi_{0y}}{\xi_{0x}} = \sqrt{\frac{r_\beta}{r_\varepsilon}} = \frac{r_\beta}{r} \tag{2.9}$$

An expression for the nominal luminosity is found by noting that

$$\xi_{0y}\left(\frac{N\gamma}{\beta_y^*}\right)_+ = \xi_{0y}\left(\frac{N\gamma}{\beta_y^*}\right)_- = \frac{r_0 N_+ N_-}{2\pi\sigma_{0y}^*\left(\sigma_{0x}^* + \sigma_{0y}^*\right)} \tag{2.10}$$

(there is, of course, a corresponding expression with $x \leftrightarrow y$). By comparing with (2.2) one finds

$$\mathscr{L}_0 = \frac{f_c}{2r_0}(1+r)\xi_{0y}\left(\frac{\gamma N}{\beta_y^*}\right)_{+,-} \equiv K(1+r)\xi_{0y}\left(\frac{\gamma N}{\beta_y^*}\right)_{+,-} \tag{2.11}$$

where the subscript +,– means that the expression in parentheses can be taken from either beam, on account of (2.4). The constant K is

$$\begin{aligned} K &= \frac{1}{2er_0 mc^2} = \frac{1}{2e^3} \\ &= 2.167\times10^{34} \quad \left[\mathrm{cm}^{-2}\mathrm{s}^{-1}\right]\cdot\left[\frac{\mathrm{cm}}{\mathrm{GeV\ A}}\right] \end{aligned} \tag{2.12}$$

where mc^2 is the rest energy of the electron and e its charge. Therefore, if we agree to express the energy E in [GeV], the current I in [A] and the beta-function in [cm], we obtain

$$\mathscr{L}_0 = 2.167\times10^{34}(1+r)\xi_{0y}\left(\frac{EI}{\beta_y^*}\right)_{+,-} \quad \left[\mathrm{cm}^{-2}\mathrm{s}^{-1}\right] \tag{2.13}$$

It should be noted that the parameters in this equation are not completely free, since they are related by Eqs. (2.8) and (2.9).

(B) Horizontal-Vertical Symmetry

By equating $\xi_{0x,+}=\xi_{0y,+}$ and $\xi_{0x,-}=\xi_{0y,-}$ one immediately finds

$$\frac{\beta^*_{x,+}}{\sigma^*_{0x}} = \frac{\beta^*_{y,+}}{\sigma^*_{0y}} \quad \text{and} \quad \frac{\beta^*_{x,-}}{\sigma^*_{0x}} = \frac{\beta^*_{y,-}}{\sigma^*_{0y}} \tag{2.14}$$

or, equivalently,

$$\left(\frac{\beta^*_y}{\beta^*_x}\right)_+ = \left(\frac{\beta^*_y}{\beta^*_x}\right)_- = \frac{\sigma^*_{0y}}{\sigma^*_{0x}} \tag{2.15}$$

By using the definition of σ_0 one concludes again that there is a single emittance ratio,

$$\left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_+ = \left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_- \tag{2.16}$$

By combining these last two equations, one obtains

$$r = r_\beta = r_\varepsilon \tag{2.17}$$

or, explicitly,

$$\left(\frac{\beta^*_y}{\beta^*_x}\right)_+ = \left(\frac{\beta^*_y}{\beta^*_{x,-}}\right)_- = \left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_+ = \left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_- = \frac{\sigma^*_{0y}}{\sigma^*_{0x}} \tag{2.18}$$

Now, by taking the ratios of the beam-beam parameters, we find

$$\frac{\xi_{0,+}}{\xi_{0,-}} = \frac{\left(N\gamma/\beta^*_x\right)_-}{\left(N\gamma/\beta^*_x\right)_+} = \frac{\left(N\gamma/\beta^*_y\right)_-}{\left(N\gamma/\beta^*_y\right)_+} \tag{2.19}$$

From this and the definition of ξ_0 one finds

$$\left(\xi_0 \frac{N\gamma}{\beta^*_y}\right)_+ = \left(\xi_0 \frac{N\gamma}{\beta^*_y}\right)_- = \frac{r_0 N_+ N_-}{2\pi\sigma^*_{0y}\left(\sigma^*_{0x} + \sigma^*_{0y}\right)} \tag{2.20}$$

with a corresponding equation with $x \leftrightarrow y$. From this and Eq. (2.2) one finds that the luminosity is

$$\mathscr{L}_0 = K(1+r)\left(\xi_0 \frac{EI}{\beta^*_y}\right)_{+,-} \tag{2.21}$$

where K is the same constant as before, Eq. (2.12). Therefore

$$\mathscr{L}_0 = 2.167\times10^{34}(1+r)\left(\xi_0\,\frac{EI}{\beta_y^*}\right)_{+,-} \quad \left[\mathrm{cm}^{-2}\mathrm{s}^{-1}\right] \tag{2.22}$$

if the energy is expressed in [GeV], the current in [A] and the beta-function in [cm].

<u>(C) Full Symmetry</u>

By full symmetry we mean that all four beam-beam parameters are equal, $\xi_{0x,+}=\xi_{0y,+}=\xi_{0x,-}=\xi_{0y,-}\equiv \xi_0$ so that all previous equations for both kinds of symmetries must be simultaneously valid. We simply summarize the results in this case:

$$\left(\frac{\beta_y^*}{\beta_x^*}\right)_+ = \left(\frac{\beta_y^*}{\beta_{x,-}^*}\right)_- = \left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_+ = \left(\frac{\varepsilon_{0y}}{\varepsilon_{0x}}\right)_- = \frac{\sigma_{0y}^*}{\sigma_{0x}^*} \tag{2.23}$$

$$\frac{\beta_{x,+}^*}{\beta_{x,-}^*} = \frac{\beta_{y,+}^*}{\beta_{y,-}^*} = \frac{(EI)_+}{(EI)_-} \tag{2.24}$$

$$\mathscr{L}_0 = 2.167\times10^{34}(1+r)\xi_0\left(\frac{EI}{\beta_y^*}\right)_{+,-} \quad \left[\mathrm{cm}^{-2}\mathrm{s}^{-1}\right] \tag{2.25}$$

where the energy is expressed in [GeV], the current in [A] and the beta-function in [cm].

3. Discussion

The main formulas in this note are those for the luminosity in terms of the beam-beam parameter, Eqs. (2.13), (2.22) and (2.25). Although we have written these in terms of vertical quantities, they can be rewritten in terms of horizontal quantities by simultaneously replacing $y \to x$ and $r \to 1/r$.

In addition to the luminosity formulas, there are several other equalities obeyed by the beta-functions and beam parameters in each case. In all cases, however, we there is a single (rather than two) beam aspect ratio r, a single emittance ratio r_ε, and a single beta-function ratio r_β. For the transparency symmetry case these three ratios may be different, although they are related by Eq. (2.8). For the horizontal-vertical symmetry, the three ratios are equal.

The ratios of beam-beam parameters are given by Eqs. (2.9) and (2.19) for the two symmetries discussed.

4. Acknowledgements

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5.References

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